JOURNAL OF APPROXIMATION THEORY 25, 193–200 (1979)

Saturation Theorems for Bernstein Polynomials in Banach Subspaces of $C(I, \mathbb{R})$

AXEL GRUNDMANN

Abteilung Mathematik, Universität Dortmund, Postfach 500 500, D-4600 Dortmund 50, West Germany

Communicated by R. Bojanic

Received July 28, 1977

1. INTRODUCTION

Let $I = [0, 1]$ be the closed unit interval. Let $C = C(I, \mathbb{R})$ be the space of all continuous functions $f: I \to \mathbb{R}$. If $\Omega \in C$ is a nonnegative and concave function, we introduce the norm $\|f\|_\Omega = \sup_{x \in I^0}(|f(x)|/\Omega(x))$, where I^0 is the open unit intervall. Now let C_Ω be the space of all $f \in C$ with $\|f\|_\Omega < +\infty$. $(C_\Omega, \|\cdot\|_\Omega)$ is a Banach subspace of C and is continuously embedded in C. It is easy to verify, if $B_n(f)$ is the n-th Bernstein polynomial, that the following hold:

$$B_n: C_\Omega \to C_\Omega,$$

$$\|B_n(f)\|_\Omega \leqslant \|f\|_\Omega,$$

$$\sup_{n \in \mathbb{N}} \|B_n\|_\Omega \leqslant 1.$$

The aim of the present paper is to characterize the class of functions $D(\theta)$, $\theta \in (0, 1]$. Here $f \in D(\theta)$ iff

$$\|B_n(f) - f\|_\Omega = O(n^{-\theta}).$$

Let D be the domain of $(B_n)_{n \in \mathbb{N}}$ in C_Ω, that is, the space of all functions $f \in C_\Omega$ with $\|B_n(f) - f\|_\Omega \to 0$. The following statements are equivalent:

$$f \in D \tag{1.1}$$

and

$$f \in \overline{C^{(2)} \cap C_\Omega}, \tag{1.2}$$

0021-9045/79/030193-08$02.00/0
Copyright © 1979 by Academic Press, Inc.
All rights of reproduction in any form reserved.

where $C^{(2)}$ is the space of all twice continuously differentiable functions in C and $-$ is the closure operator in C_Ω. In general $C \neq C_\Omega$. If we assume

$$\lim_{x\to 0} \frac{x}{\Omega(x)} = 0 = \lim_{x\to 1} \frac{x}{\Omega(x)},$$

then it is easy to prove that D consists of all $f \in C_\Omega$ for which

$$\lim_{x\to 0} \frac{f(x)}{\Omega(x)} = 0 = \lim_{x\to 1} \frac{f(x)}{\Omega(x)}.$$

In this case $D \neq C_\Omega$ ($f = \Omega$).

2. Inequalities

For $0 \leqslant h \leqslant x \leqslant 1 - h \leqslant 1$ and $f\colon I \to \mathbb{R}$ we set:

$$\Delta_h^2 f(x) = f(x+h) - 2f(x) + f(x-h).$$

Let $L_1(x) = x\log(x)$, $L_2(x) = L_1(1-x)$ and $L = L_1 + L_2$.

In the following lemma we collect several simple inequalities needed in this paper.

LEMMA (2.0). *We have*

$$B_n(L_1)x - L_1(x) \leqslant \frac{1-x}{n}, \tag{2.1}$$

$$B_n(L_2)x - L_2(x) \leqslant \frac{x}{n}, \tag{2.2}$$

$$\Delta_h^2 L_1(x) \leqslant 2\frac{h^2}{x}, \tag{2.3}$$

$$\Delta_h^2 L_2(x) \leqslant 2\frac{h^2}{1-x}, \tag{2.4}$$

$$B_n(L_1)''\, x \leqslant \frac{2}{x}, \tag{2.5}$$

$$B_n(L_2)''\, x \leqslant \frac{2}{1-x}. \tag{2.6}$$

Proof. For (2.1):

$$B_n(L_1)x = \sum_{k=0}^{n} \frac{k}{n} \log\left(\frac{k}{n}\right) p_{k,n}(x),$$

$$B_n(L_1)x - L_1(x) = \sum_{k=0}^{n} \frac{k}{n} \left(\log\left(\frac{k}{n}\right) - \log(x)\right) p_{k,n}(x).$$

Since log is concave, we have the inequality

$$\log\left(\frac{k}{n}\right) - \log(x) \leqslant \frac{1}{x}\left(\frac{k}{n} - x\right).$$

Then

$$\begin{aligned} B_n(L_1)x - L_1(x) &\leqslant \frac{1}{x} \sum_{k=0}^{n} \frac{k}{n}\left(\frac{k}{n} - x\right) p_{k,n}(x) \\ &= \frac{1-x}{n}. \end{aligned}$$

(2.2) is a direct consequence of (2.1).

For (2.3) and (2.4) see [1], (2.5) is by follows from

$$\begin{aligned} B_n(L_1)'' x &= n(n-1) \sum_{k=0}^{n-2} \Delta_{1/n}^2 L_1\left(\frac{k+1}{n}\right) p_{k,n-2}(x) \\ &\leqslant n(n-1) \sum_{k=0}^{n-2} 2\, \frac{(1/n)^2}{(k+1)/n}\, p_{k,n-2}(x) \\ &\leqslant 2(n-1) \sum_{k=0}^{k-2} \frac{p_{k,n-2}(x)}{k+1}. \end{aligned}$$

Since

$$\frac{1}{k+1} = \int_0^1 \xi^k \, d\xi,$$

we have

$$\begin{aligned} B_n(L_1)'' x &\leqslant 2(n-1) \int_0^1 (1 - x + \xi x)^{n-2} \, d\xi \\ &= \frac{1-(1-x)^{n-1}}{x} \\ &\leqslant \frac{2}{x}; \end{aligned}$$

Q.E.D.

The last two inequalities are direct consequences of the fact that Ω is a concave function.

LEMMA (2.7). *Let* $\Phi(x) = x(1-x)$. *Then*

$$\frac{n}{\Phi(x)} \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 \Omega\left(\frac{k}{n}\right) P_{k,n}(x) \leqslant \Omega\left(x + \frac{1-2x}{n}\right), \tag{2.8}$$

$$\frac{n}{(n-1)\,\Phi(x)} \sum_{k=0}^{n} \Phi\left(\frac{k}{n}\right) \Omega\left(\frac{k}{n}\right) p_{k,n}(x) \leqslant \Omega\left(x + \frac{1-2x}{n}\right). \tag{2.9}$$

Proof. Follows from the fact that the left sides of (2.8) and (2.9) are convex combinations.

3. A K-FUNCTIONAL FOR $\| B_n(f) - f \|_\Omega$.

If $g \in C^{(2)}$ and $k(x, u)$ is Green's kernel for the differential operator $l(g) = g''$, with $g(0) = g(1) = 0$, then we have

$$g(x) = a(x) + \int_0^1 k(x, u)\, g''(u)\, du,$$

where a is a linear function and

$$k(x, u) = \begin{cases} (x-1)u, & 0 \leqslant u \leqslant x, \\ x(u-1), & x \leqslant u \leqslant 1. \end{cases}$$

Now we define a K-functional for $f \in C_\Omega$. If $t \geqslant 0$, and $A = C^{(2)} \cap C_\Omega$, we set:

$$K(t, f) = \inf_{g \in A} (\| f - g \|_\Omega + t \| \Phi g'' \|_\Omega).$$

LEMMA (3.0). *If* $f \in C_\Omega$, *then*

$$\| B_n(f) - f \|_\Omega \leqslant 3K\left(\frac{1}{n}, f\right).$$

Proof. First, it is clear that $\| B_n(f) - f \|_\Omega \leqslant 2 \| f \|$. Secondly, we have to estimate $\| B_n(g) - g \|_\Omega$ for $g \in A$. Since

$$B_n(g)x - g(x) = \int_0^1 (B_n(k(\cdot, u))x - k(x, u))\, g''(u)\, du,$$

$$| B_n(g)x - g(x) | \leqslant \| \Phi g'' \|_\Omega \int_0^1 (B_n(k(\cdot, u))x - k(x, u)) \frac{\Omega(u)}{\Phi(u)}\, du.$$

Note that $L_2(x) = \int_0^1 (k(x, u)/(1 - u))\, du$. Also $\Omega(u) - \Omega(x) \leqslant \Omega^{[1]}(x)(u - x)$ where $\Omega^{[1]}(x) = \frac{1}{2}(\Omega'(x - 0) + \Omega'(x + 0))$ for $x \in I^0$, since Ω is concave. Moreover, we have as an essential consequence of the fact that Ω is a concave function, the inequality $| \Omega^{[1]}(x)|\, \Phi(x) \leqslant \Omega(x)$. By all this it is easy to verify the following estimates:

$$\begin{aligned}\int_0^1 (B_n(k(\cdot, u))x - k(x, u)) \frac{\Omega(u)}{\Phi(u)}\, du & \\ \leqslant \Omega(x)(B_n(L)x - L(x)) &+ | \Omega^{[1]}(x)| \, ((1 - x)(B_n(L_2)x - L_2(x)) \\ &+ x(B_n(L_1)x - L_1(x))) \\ \leqslant \frac{1}{n} \Omega(x) + | \Omega^{[1]}(x)| \frac{2\Phi(x)}{n} & \\ \leqslant \frac{3}{n} \Omega(x). &\end{aligned}$$

(see (2.1) and (2.2)). The rest follows by standard arguments.

4. Inverse Arguments

We follow the main ideas of [1]. Thus, we have to estimate $\| \Phi B_n(f)'' \|_\Omega$ for $f \in C_\Omega$ and for $f \in A$.

Lemma (4.0). *If $f \in C_\Omega$, then*

$$\| \Phi B_n(f)'' \|_\Omega \leqslant 16n \| f \|_\Omega .$$

Proof.

$$B_n(f)''\, x = n(n - 1) \sum_{k=0}^{n-2} \Delta_{1/n}^2 f\left(\frac{k + 1}{n}\right) p_{k,n-2}(x).$$

Since Ω is concave,

$$| B_n(f)''\, x | \leqslant 4n^2 \| f \|_\Omega\, \Omega\left(\frac{n - 2}{n} x + \frac{1}{n}\right)$$

and by the proof of Lemma (3.0),

$$\leqslant 4n^2 \| f \|_\Omega \left\{\Omega(x) + | \Omega^{[1]}(x)| \frac{3}{n}\right\}.$$

Thus we have

$$\begin{aligned}| \Phi(x)\, B_n(f)''\, x | &\leqslant 4n^2 \left\{\Phi(x) + \frac{3}{n}\right\} \Omega(x) \| f \|_\Omega \\ &\leqslant 4n\{n\Phi(x) + 3\}\, \Omega(x) \| f \|_\Omega .\end{aligned}$$

For $n\Phi(x) \leqslant 1$ it follows that $| \Phi(x)\, B_n(f)''\, x | \leqslant 16n \| f \|_\Omega\, \Omega(x)$.

We have

$$\Phi(x)\, B_n(f)''\, x = \frac{n}{\Phi(x)} \sum_{k=0}^{n} \left\{(n-1)\left(\frac{k}{n}-x\right)^2 - \frac{k}{n}\left(1-\frac{k}{n}\right)\right\} f\left(\frac{k}{n}\right) p_{k,n}(x).$$

Obviously, see (2.8) and (2.9),

$$\begin{aligned} |\,\Phi(x)\, B_n(f)''\, x\,| &\leqslant \|f\|_\Omega\{(n-1)+(n-1)\}\, \Omega\left(x + \frac{|\,1-2x\,|}{n}\right) \\ &\leqslant 2(n-1)\, \|f\|_\Omega \left(\Omega(x) + |\,\Omega^{[1]}(x)|\, \frac{|\,1-2x\,|}{n}\right) \\ &\leqslant 2n\, \|f\|_\Omega \left(1 + \frac{1}{n\Phi(x)}\right) \Phi(x). \end{aligned}$$

If now $n\Phi(x) \leqslant 1$, we get

$$|\,\Phi(x)\, B_n(f)''\, x\,| \leqslant 4n\, \|f\|_\Omega\, \Omega(x),$$

and Lemma (4.0) is proved.

LEMMA (4.1). *If $g \in A$, then*

$$\|\,\Phi B_n(g)''\,\|_\Omega \leqslant 6\, \|\,\Phi g''\,\|_\Omega\,.$$

Proof. By applying the representation

$$g(x) = \int_0^1 k(x, u)\, g''(u)\, du + a(x),$$

we have

$$B_n(g)''\, x = \int_0^1 B_n(k(\cdot, u))''\, x g''(u)\, du.$$

Hence

$$\begin{aligned} |\,B_n(g)''\, x\,| &\leqslant \|\,\Phi g''\,\|_\Omega \int_0^1 B_n(k(\cdot, u))''\, x\, \frac{\Omega(u)}{\Phi(u)}\, du \\ &\leqslant \|\,\Phi g''\,\|_\Omega\{\Omega(x)\, B_n(L)''\, x \\ &\qquad + |\,\Omega^{[1]}(x)|\, ((1-x)\, B_n(L_2)''\, x + x B_n(L_1)''\, x)\} \\ &\leqslant \|\,\Phi g''\,\|_\Omega\, \Omega(x) \left(\frac{2}{\Phi(x)} + \frac{4}{\Omega(x)}\right), \end{aligned}$$

and the result follows.

5. THE FUNCTION CLASS $D(\theta)$

THEOREM (5.0). *For* $0 < \theta < 1$ *we have*

$$\| B_n(f) - f \| = O(n^{-\theta})$$

iff

$$K(t, f) = O(t^{\theta}).$$

Proof. By use of the main ideas of [1].

We give now another characterization of the class $D(\theta)$ using Theorem (5.0). This characterization is less implicit than the one by the K-functional. Let L be a linear continuous functional over C with the property that the linear functions are in the kernel of L. By Riesz's representation theorem we have

$$L(f) = \int_0^1 f(x)\, d\mu(x).$$

We now define

$$| L | (f) = \int_0^1 f(x) \mid d\mu(x)|.$$

For $f \in C_\Omega$,

$$| L(f)| \leqslant \|f\|_\Omega \, | L | (\Omega).$$

For $g \in C^{(2)} \cap C_\Omega$, $g(x) = l(x) + \int_0^1 k(x, u)\, g''(u)\, du$ where l is a linear function,

$$| L(g)| \leqslant \| \Phi g'' \|_\Omega \int_0^1 | L(k(\cdot, u))| \frac{\Omega(u)}{\Phi(u)}\, du.$$

Then, for $f \in C_\Omega$,

$$| L(f)| \leqslant | L | (\Omega) \cdot K \left\{ \frac{1}{| L | (\Omega)} \int_0^1 | L(k(\cdot, u))| \frac{\Omega(u)}{\Phi(u)}\, du, f \right\}.$$

Let C_0^* be the space of all linear continuous functionals L over the space C, with the property that $L(\phi) = 0$ whenever ϕ is a linear function.

LEMMA (5.1). *For* $0 < \theta < 1$ *the following statements* (i) *and* (ii) *are aquivalent*:

(i) $f \in D(\theta)$.

(ii) *For all* $L \in C_0^*$,

$$| L(f)| \leqslant M_f \{| L | (\Omega)\}^{1-\theta} \left\{ \int_0^1 | L(k(\cdot, u))| \frac{\Omega(u)}{\Phi(u)}\, du \right\}^{\theta}$$

Proof. (i) $\Rightarrow$ (ii) is trivial: see the above arguments.

(ii) $\Rightarrow$ (i): Let us define a special $L \in C_0^*$, for a fixed $x \in I$, by

$$L(f) = B_n(f)x - f(x).$$

Then

$$|L|(\Omega) \leqslant \Omega(x) + B_n(\Omega)x \leqslant 2\Omega(x),$$

$$\int_0^1 |L(k(\cdot, u))| \frac{\Omega(u)}{\Phi(u)} du = B_n(F)x - F(x)$$

where

$$F(x) = \int_0^1 k(x, u) \frac{\Omega(u)}{\Phi(u)} du,$$

and, see Lemma (3.0),

$$B_n(F)x - F(x) \leqslant c \frac{\Omega(x)}{n}; \qquad \text{Q.E.D.}$$

Remark. Lemma (5.1) is also valid for $\theta = 1$. For the proof we have to modify the methods of [2].

References

1. H. Berens and G. G. Lorentz, Inverse theorems for Bernstein polynomials, *Indiana Univ. Math. J.* **21** (1972), 693–708.
2. G. G. Lorentz, Inequalities and saturation classes of Bernstein polynomials, in "On Approximation Theory," pp. 200–207, Proc. Conference Oberwolfach 1963, Birkhäuser, Basel/Stuttgart, 1964.